The Gini Coefficient and Personal Inequality Measurement

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Abstract

The Gini coefficient is based on the sum of pairwise income differences, which can be decomposed into separate sums for individuals. Differences vis-à-vis poorer people represent an individual's *advantage*, while those with respect to richer people constitute *deprivation*. Weighting deprivation and advantage differently produces a family of "Gini admissible" personal inequality indexes, whose population averages each equal the Gini coefficient. Properties of the personal indexes illuminate those of the Gini. Secular changes in income distribution are analyzed. During economic development, people in the traditional sector may view inequality as constantly increasing while those in the modern sector believe the opposite. Personal inequality assessments during periods of polarization and rising overall inequality are also discussed.

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I. Introduction

The Gini coefficient has a natural interpretation as the mean of personal inequality assessments. While that fact is fairly obvious, it was not emphasized in the original work by Gini (1914) and has not been highlighted since. This paper shows that this straightforward interpretation throws important light on the properties of the Gini coefficient. It also allows us to better understand individual reactions, as well as that of the Gini coefficient, to secular changes in income distribution. The latter include the transition from a traditional to a modern economy analyzed by Kuznets (1955), and the polarization and rising inequality seen in recent decades in the U.S. and many other countries. Personal assessments of even the direction of change in inequality may differ between people at different income levels. These results suggest that our understanding of inequality measurement can be enriched by studying what it may mean at the personal level.

The Gini coefficient can be defined or interpreted in many ways (Yitzhaki, 1998). For our purposes the most useful is that it equals one half the mean difference divided by the mean. For a finite population, the Gini coefficient can be found by taking the sum of all all pairwise absolute income differences, S, converting to an average and normalizing by the mean. S can be written as the sum across individuals $i = 1, .., n$ of their individual sums of pairwise differences with all other individuals, S_i. The latter can be used as the basis for a personal inequality index whose average across the population is the Gini coefficient. For each individual, S_i is composed of the sum of differences with higher incomes plus the sum of differences with lower incomes. Following Yitzhaki (1979) the sum of differences with higher incomes may be used to define the individual's *deprivation*. That concept is complemented by the individual's *advantage,* derived from the sum of differences with respect to lower incomes.[1] Summing deprivation or advantage across the whole population produces the same total (Yitzhaki, 1979). An implication is that a *weighted* average of deprivation and advantage, as well as an unweighted average, will generate a personal inequality index that will equal the Gini coefficient when averaged across the population. This means that there is a whole family of "Gini admissible" personal inequality indexes or GAPIIs. If societies choose to base overall inequality measurement on an average of individual assessments they may all use the same inequality index, that is the Gini coefficient, at the aggregate level even if they differ in the weight their members place on advantage vs. deprivation. [2]

The personal inequality indexes discussed here may be regarded from a "top down" or "bottom up" viewpoint. A GAPII could be interpreted as showing how a social planner would measure inequality at the personal level. This is a "top down" view. An alternative, "bottom up", view is that *individuals*, for whatever reason, assess inequality using a GAPII. Why might individuals do so? One possibility is that they could have interdependent utility functions, such as that proposed by Fehr and Schmidt (1999), which suggest the use of a GAPII. But one may also appeal to bounded rationality. The difference

[1] Yitzhaki (1979) used the term "relative deprivation", which was introduced by Runciman (1966) to refer to any case in which some members of a reference group felt deprived compared to other members of their group. "Deprivation" is used here simply because it is shorter. Fehr and Schmidt (1999) referred to the same concept as "disadvantageous inequality", but the term deprivation still dominates in the literature. Yitzhaki (1979) used the term "satisfaction" rather than "advantage". "Advantage" is used here as a more neutral term.

[2] It may seem too strong to assume that all individuals in a society would place the same weight on advantage vs. deprivation. With continuous income distributions this assumption could be relaxed to allow weights to differ across individuals as long as those differences were independent of income.

between incomes is an unobjectionable indicator of inequality between two people (especially if considered in the light of mean income). Although we know there are many alternatives, people may simply think, by extension, that the average of such differences provides the natural basis for measuring inequality when there are more than two people. That conclusion could be reinforced by information and computing constraints. As shown in this paper, in order to compute the value of a GAPII the individual only needs to know the fraction of the population with income above him and the average incomes of those above and below him. While it is not reasonable to suppose that each individual knows everyone's income, he/she might be able to make a serviceable guess at these three quantities.

This paper is related to the large literature on individual attitudes toward inequality. A portion of the literature attempts to measure attitudes within narrow reference groups, e.g. co-workers or members of the same occupation. In that context people tend to be averse to deprivation but to like advantage. As Clark and D'Ambrosio (2015) point out, in the income distribution literature the usual reference group is broader. In that context, following Yitzhaki (1979, 1982) and Fehr and Schmidt (1999) the general expectation has been that people will be averse to both deprivation and advantage. There are now a few empirical and experimental studies that have estimated aversion to deprivation and/or advantage with broader reference groups. Using the German SOEP survey data, D'Ambrosio and Frick (2007) find strong aversion to deprivation (but do not report on attitudes to advantage). Cojocaru (2014) finds significant aversion to both advantage and deprivation using a survey of 27 transition countries. In experiments with subjects who played a sequential public goods game, Teyssier (2012) found that 40% were averse to both advantage and deprivation while 18% were averse to neither. While these studies do not indicate a difference in aversion to deprivation vs. advantage, neural studies find that brain activity reacts more strongly to deprivation and some authors presume that aversion to advantage is likely weaker than aversion to deprivation (Clark and D'Ambrosio, 2015).

The remainder of the paper proceeds as follows. For expositional simplicity we start by working with the case in which advantage and deprivation are equally weighted. Section II defines the personal inequality index and derives some of its basic properties. In Section III we then explore how the behavior of this index helps to explain the sensitivity of the Gini coefficient to income changes in different ranges of a distribution. The analysis is extended to allow unequal weighting of deprivation and advantage in Section IV, which shows that the main insights of the previous two sections survive this generalization. How the personal assessments of inequality vary with income is discussed in Section V and the behavior of those assessments during period of secular change in income distribution is examined in Section VI. Section VII concludes.

II. Gini-admissible Personal Inequality Indexes: Base Case

In this section we see how the Gini coefficient can be defined as the average value across individuals of a particular personal inequality index (PII), and begin to examine the properties of the latter. We do not seek a basis for the PII in individuals' personal or social preferences. Our interest is confined to investigating the implications for personal inequality assessments if individuals use a Gini admissible PII, or GAPII. A PII will be termed Gini admissible if the Gini coefficient can be found by taking a simple average of the values of that PII across individuals.

The Gini coefficient for an income distribution equals one half the mean difference divided by the mean, as we see in:

$$(1) \qquad G = \frac{1}{2n^2\bar{y}}\sum_{i=1}^{n}\sum_{j=1}^{n}|y_i - y_j| = \frac{S}{2n^2\bar{y}}$$

where y_i is the income of individual i, $\bar{y}$ is mean income, $n > 1$, $y_1 \leq y_2 \leq \cdots \leq y_n$, S is the sum of differences, and S/n^2 is the mean difference.[3]

A natural but previously overlooked interpretation is that G is the mean value across individuals of a particular GAPII, G_i:

$$(2) \qquad G = \frac{1}{n}\sum_{i=1}^{n} G_i$$

where

$$(3) \qquad G_i = \frac{1}{2n\bar{y}}\sum_{j=1}^{n}|y_i - y_j| = \frac{S_i}{2n\bar{y}}$$

and S_i is the sum of differences for individual i. Equation (3) can be rewritten:

$$(4) \qquad G_i = \frac{1}{2n\bar{y}}\left[n_i^l(y_i - \bar{y}_i^l) + n_i^h(\bar{y}_i^h - y_i)\right]$$

where n_i^l is the number of individuals with income less than or equal to y_i, excluding individual i, and n_i^h is the number with income strictly greater than y_i, so that $n_i^l + n_i^h = n - 1$.[4] $\bar{y}_i^l$ and $\bar{y}_i^h$ are mean income among those with income less than or equal to y_i, excluding i, and strictly greater than y_i respectively.

Let H_i be the set of all j such that $y_j > y_i$, and L_i be the set of all j excluding i such that $y_j \leq y_i$. Equation (4) can be expressed as:

$$(4') \qquad G_i = \frac{1}{2\bar{y}}(A_i + D_i)$$

where:

$$(5i) \qquad A_i = \frac{n_i^l}{n}\left(y_i - \bar{y}_i^l\right) = \frac{1}{n}\sum_{j \in L_i}(y_i - y_j)$$

[3] As mentioned earlier, the Gini coefficient can be expressed in many different ways (Yitzhaki, 1998). This is one of the two principal forms in which it was originally set out in Gini (1914), and is the most convenient for our discussion.

[4] The choice to include individuals who have the same income as i in the lower group rather than in the higher group is arbitrary but does not affect the results in any significant way.

$$(5ii) \quad D_i = \frac{n_i^h}{n}\left(\bar{y}_i^h - y_i\right) = \frac{1}{n}\sum_{j\in H_i}(y_j - y_i)$$

D_i is the discrete analogue of the measure of relative deprivation for an individual, which we will refer to simply as deprivation, proposed by Yitzhaki (1979) for a continuous distribution. It equals the average shortfall of *i*'s income below the income of those who are better off, weighted by the fraction of the population in the latter group. Equation (4′) shows that $G_i$is the simple average of D_i and a complementary measure, A_i, normalized by the mean. We will say that A_i represents individual *i*'s *advantage* compared to people with lower income. Thus from the individual perspective inequality consists of both deprivation with respect to the better off and advantage over the worse off.

While G_i is a natural personal inequality index to associate with the Gini coefficient, it is not the only GAPII. As mentioned earlier, and as shown in Section IV, one can define a more general class of GAPIIs that are based on a weighted average of A_i and D_i. G_i is a special case in which the weights on A_i and D_i are equal.

From (4) we have:

Proposition 1: *G_i is insensitive to a transfer of income within H_i or within L_i if the composition of neither group changes as a result of the transfer.*

The proposition follows from the fact that transfers of income confined either to H_i or L_i do not alter n_i^l, $\bar{y}_i^l$, n_i^h, or $\bar{y}_i^h$ or any other term on the right-hand side of (4). In terms of (4′), as noted by Yitzhaki (1979) these transfers have no effect on advantage, A_i, or on deprivation, D_i. The insensitivity of G_i to such transfers means that it does not respect the Pigou-Dalton principle of transfers, which is a cornerstone of the theory of aggregate inequality measurement. That an aggregate index that respects the Pigou-Dalton principle can be built on the basis of personal indexes that violate the principle is striking.

Sensitivity of G_i to a transfer of income between H_i and L_i

What determines how sensitive G_i is to a transfer of income between H_i and L_i? Consider the transfer of a total amount *R* from H_i to L_i. Note that such a transfer reduces both $A_i$and D_i by *R/n*, as can be seen from (5) where $n_i^l(y_i - \bar{y}_i^l)$ and $n_i^h(\bar{y}_i^h - y_i)$ both fall by *R*. We will allow *R* to be negative, so this also handles the case of transfers from L_i to H_i, which *increase* A_i and D_i by equal amounts. Using

$$\frac{\partial A_i}{\partial R} = \frac{\partial D_i}{\partial R} = \frac{-1}{n}$$

from (4′) we have:

$$(6) \quad \frac{\partial G_i}{\partial R} = -\frac{1}{n\bar{y}}$$

which allows us to state:

Proposition 2: When income is transferred from a person with income strictly above y_i to someone with income strictly below y_i, G_i falls, while if income is transferred from a person with income strictly below

y_i to someone with income strictly above y_i, G_i rises. In both cases the change in G_i is proportional to the amount transferred and independent of y_i.

Note that this proposition implies that any given individual is equally sensitive to a transfer from the group above him to the group below, or vice versa. In that sense, individuals are equally sensitive to redistribution that does not alter their own income.

Sensitivity of G_i to a transfer affecting y_i

We also need to analyze those cases where distributional changes affect individual *i*'s own income. There are two situations to consider. One is that of a transfer from *i* to another person *j*. The other is that of a transfer from *j* to *i*. We will consider them in turn. *In this analysis, and in the remainder of the paper unless indicated otherwise, we will assume $y_1 < y_2 < \cdots < y_n$.* This assumption will simplify the analysis since, for example, it implies that when *n* is odd there is a unique individual with median income, y^{med}, and half the remaining population has $y_i < y^{med}$ while the other half have $y_i > y^{med}$.[5] If *n* is even there is no individual with $y_i = y^{med}$, but y^{med}, which is defined as the midpoint between $y_{n/2}$ and $y_{n/2+1}$, again divides the population into two sub-populations of equal size with incomes above and below the median.

Transfer from *i* to *j*: Let y_i^o and y_j^o be initial incomes and consider the effect on G_i of the transfer of a small amount *r* from individual *i* to individual *j*. From (4) we obtain:

Proposition 3a: The effect on G_i of a small transfer in the amount of *r* from individual *i* to an individual *j* is given by:

$$(7i) \qquad \Delta G_i = \frac{1}{2n\bar{y}}\left[\left(n_i^h - n_i^l\right) - 1\right]\mathrm{r}\,, \qquad i > j$$

$$(7ii) \qquad \Delta G_i = \frac{1}{2n\bar{y}}\left[\left(n_i^h - n_i^l\right) + 1\right]\mathrm{r}\,, \qquad i < j$$

If we could ignore the -1 and +1 in the square brackets on the right-hand side, (7) would say that irrespective of whether *i* was greater or less than *j*, a transfer from *i* to anyone else would increase G_i if *i* was below the median and reduce G_i if *i* was above the median. This reflects the fact that the main impact of the transfer on G_i is to reduce A_i and increase D_i. If $n_i^h > n_i^l$, individual *i* is below the median and from (5) we see that the increase in D_i will exceed the drop in A_i, since those changes are proportional to n_i^h and n_i^l respectively. If $n_i^h < n_i^l$, individual *i* is above the median and we have the opposite case. The -1 in (7i) means that the rank at which ΔG_i switches from being positive to negative as we go up the income scale in the $i > j$ case is one position higher than it would otherwise be, since the transfer is going to a person with income lower than the "donor" *i*, which reduces $\bar{y}_i^l$ and A_i a little. And

[5] If we assume only $y_1 \leq y_2 \leq \cdots \leq y_n$ then there could be multiple individuals with median income and the groups with income strictly below the median and strictly above the median need not contain an equal number of members. Consider for example a population with the set of incomes (1, 1, 2, 2, 2, 3).

the +1 in (7ii) means that when $i < j$, ΔG_i switches from positive to negative one position *lower* than would otherwise be the case since the transfer goes to a higher income person, raising $\bar{y}_i^h$ and D_i a little.

Transfer from *j* to *i*: Here incomes after a transfer are $y_i^o + r$ and $y_j^o - r$. and we have:

Proposition 3b: The effect on G_i of a small transfer in the amount of *r* from an individual *j* to individual *i* is given by:

$$(8i) \qquad \Delta G_i = \frac{1}{2n\bar{y}}\left[\left(n_i^l - n_i^h\right) + 1\right]\mathrm{r}\,, \qquad i > j$$

$$(8ii) \qquad \Delta G_i = \frac{1}{2n\bar{y}}\left[\left(n_i^l - n_i^h\right) - 1\right]\mathrm{r}\,, \qquad i < j$$

Now the main effect of the transfer is to raise y_i and therefore to increase A_i and reduce D_i, which is equalizing if y_i is below the median and disequalizing if y_i is above the median. Again the point at which ΔG_i switches sign as *i* rises is offset one position by the small impact of the change in y_j on A_i when $i > j$ and on D_i when $i < j$.

Summing up, we can say, somewhat loosely, that an individual perceives a small transfer from himself to someone else as equalizing if his income is above the median, and as disequalizing if his income is below the median. If he is the *recipient* he finds a small transfer equalizing if he is below the median and disequalizing if he is above the median. Thus the situation in Gini-admissible personal inequality measurement is quite different from that in the familiar aggregate inequality measurement. In the latter, the impact of a small transfer on inequality is deemed equalizing if the donor's income exceeds the recipient's and disequalizing if the opposite holds. In the case of Gini-admissible personal inequality measurement, in contrast, whether the transfer is considered equalizing or disequalizing depends almost solely on the income of the person making the assessment. Low income people find making a transfer disequalizing and receiving a transfer equalizing. High income people find the opposite.

III. Explaining the sensitivity of the Gini coefficient to changes in different ranges of the income distribution

From (1) one may derive:

$$(9) \qquad G = \frac{2}{n^2\bar{y}}[y_1 + 2y_2 + 3y_3 + \cdots + ny_n] - \frac{n+1}{n}$$

(see e.g. Cowell, 2011, p. 114). This provides insight into the sensitivity of the Gini coefficient to changes in different ranges of the income distribution. Consider a small transfer, *r*, from individual *j* to individual *i* where $i < j$. This is an example of what would be called an "equalizing transfer" in discussions of aggregate inequality. From (9), this transfer will produce a change in the Gini coefficient given by:

$$(10) \qquad \Delta G = \frac{-2r(j-i)}{n^2\bar{y}}$$

which also tells us the impact of a transfer from *i* to *j*, in which case $r < 0$. We see that the impact on the Gini coefficient does not depend on y_i or y_j, but varies only with *r* and the difference in income ranks between *i* and *j*.

The fact that the sensitivity of the Gini coefficient to transfers is independent of the incomes of the transferor and transferee, but depends on the number of people between them in the distribution is one of the most interesting properties of the Gini coefficient. This property follows directly from those of the personal inequality index G_i captured in Propositions 1, 2 and 3 above. Again considering a small transfer, *r*, from individual *j* to individual *i* where *i* < *j* , Proposition 1 implies:

(11i) $\quad \Delta G_k = 0. \quad k < i, k > j.$

From Proposition 2 we have:

(11ii) $\quad \Delta G_k = \frac{-r}{n\bar{y}}. \quad i < k < j.$

And from Proposition 3

(12) $\quad \Delta G_i = \frac{(n_i^l - n_i^h - 1)r}{2n\bar{y}}. \qquad \Delta G_j = \frac{(n_j^h - n_j^l - 1)r}{2n\bar{y}}$.

Now, from (2) and (11i), the change in *G* resulting from a transfer from *j* to *i* is given by:

(13) $\quad \Delta G = \frac{1}{n}(\Delta G_i + \Delta G_j + \sum_{k=i+1}^{j-1} \Delta G_k)$

Note first from (11*ii*) that

(14) $$\sum_{k=i+1}^{j-1} \Delta G_k = -(j - i - 1)\frac{r}{n\bar{y}}$$

which is proportional to the number of people between *i* and *j*, that is the number of people the transfer from *j* to *i* "passes over".

Next, to complete the analysis of ΔG, note from (12) that:

$$\Delta G_i + \Delta G_j = \frac{(n_i^l - n_i^h - 1)r}{2n\bar{y}} + \frac{(n_j^h - n_j^l - 1)r}{2n\bar{y}}$$

$$= \frac{-r}{2n\bar{y}}[(n_j^l - n_i^l) + (n_i^h - n_j^h) + 2]$$

Since $n_j^l - n_i^l$ and $n_i^h - n_j^h$ both equal *j* – *i* we have:

(15) $\quad \Delta G_i + \Delta G_j = \frac{-r}{n\bar{y}}(j - i + 1)$

Hence, like $\sum_{k=i+1}^{j-1} \Delta G_k$, $\Delta G_i + \Delta G_j$ is proportional to the size of the transfer and rises linearly, in absolute value, with the number of people between *i* and *j*.[6] In this case the reason for dependence on the number of people between *i* and *j* is that the effects of the transfer cancel out for A_i and A_j on the one hand, and for D_i and D_j on the other, where the sums they are based on overlap. The range of overlap includes all $k < i$ for A_i and A_j, and all $k > j$ for D_i and D_j. The range where effects do not cancel out has $j - i + 1$ people in it.

Summing up, substituting (14) and (15) into (13) we have:

$$(16) \qquad \Delta G = \frac{-r}{n^2 \bar{y}} [(j - i + 1) + (j - i - 1)] = \frac{-2r(j - i)}{n^2 \bar{y}}$$

So we have shown that the mean of the effects on the personal inequality indexes resulting from the transfer equals the change in *G* that one would expect from aggregate inequality analysis.

The purpose of this exercise has been to show that the effects of a transfer on personal inequality explain the impact on *G*. That the reaction of *G* is governed by the number of people between transferor *j* and transferee *i* is due to two things: (i) aside from *i* and *j* themselves, the only people who care about the transfer are the individuals between them in the distribution, and (ii) the effects of the transfer on G_i and G_j cancel out except for those based on changes in income gaps between *i* or *j* and individuals in the range (*i*+1, *j*-1).

IV. Unequal Weighting of Deprivation and Advantage

Yitzhaki (1979) defined relative deprivation for a society as a whole, *D*, as the average of individual deprivation indexes D_i. He worked with continuous distributions. The corresponding relationship with a discrete income distribution is:

$$(17) \qquad D = \frac{1}{n} \sum_{i=1}^{n} D_i$$

We can define overall advantage in a parallel way as:

$$(18) \qquad A = \frac{1}{n} \sum_{i=1}^{n} A_i$$

Yitzhaki shows that *D* is related to the Gini coefficient according to:

$$(19) \qquad G = \frac{D}{\bar{y}}$$

[6] Note that the right-hand-side of (15) is not *proportional* to the number of people between *i* and *j*, which is $j - i - 1$.

This result might appear puzzling, given that, from (4′), D_i represents only part of an individual's contribution to G_i and therefore to *G*. The explanation is as follows. The Gini coefficient is proportional to the sum of differences, *S*. We can arrange the pairwise differences $|y_i - y_j|$ making up *S* in a matrix *M* with *i* indexing rows and *j* indexing columns. *D* is the mean of the above-diagonal elements of *M* while *A* is the mean of the below-diagonal elements. Now, the above-diagonal elements have the same mean as the below-diagonal elements in *M*, since e.g. $|y_2 - y_1| = |y_1 - y_2|$. Hence *A* = *D*. To get from *D* to *S* we must therefore double *D* and multiply by n^2 (to go from an average to a sum). The same procedure could be used to generate *S* from *A*. Thus we have $S = 2n^2 D = 2n^2 A$ or:

$$(20) \qquad A = D = \frac{S}{2n^2}$$

Substituting the expression for *D* from (20) into (19) we obtain $G = S/(2n^2\bar{y})$, that is equation (1).

While Yitzhaki's approach and ours are closely related, his D_i and our G_i are distinct. G_i depends not just on deprivation, D_i, but also on advantage, A_i. While, overall, *A* = *D*, at the individual level there is no such relationship. A_i rises and D_i falls as we move up through the income distribution from y_1 to y_n, and they do so at rates that rise or fall depending on the shape of the particular income distribution being examined.

The fact that $A = D$ has important consequences for our personal inequality indexes. Using (19) and $A = D$, *G* may be found by taking a weighted average of *A* and *D*, as in:

$$(21) \qquad G = \frac{\lambda A + (1-\lambda)D}{\bar{y}} \qquad\qquad 0 \leq \lambda \leq 1$$

where we require the weights to be positive. This in turn reveals that there is a family of Gini admissible personal inequality indexes or GAPIIs of the form:

$$(22) \qquad G_i^{\lambda} = \frac{\lambda A_i + (1-\lambda)D_i}{\bar{y}} \qquad\qquad 0 \leq \lambda \leq 1$$

Hence, while λ may differ across societies, they can nevertheless agree on using *G* as an aggregate measure of inequality. In the continuous case this result could be generalized to allow λ to differ across individuals, as long as the distribution of λ was independent of individual income.

We may ask which of the results derived above for the λ = ½ case survive once $\lambda \neq$ ½ is allowed. Proposition 1, which says that the G_i are insensitive to transfers entirely within the H_i or L_i comparator groups, survives. The principle is not affected by re-weighting income differences with the H_i and L_i groups via λ≠ ½ . Proposition 2, which says that when income is transferred from those with income above (below) y_i to those with income below (above) y_i the fall (rise) in G_i is proportional to the total amount transferred, *R*, and is independent of y_i is also unaltered because we still have:

$$\frac{\partial A_i}{\partial R} = \frac{\partial D_i}{\partial R} = \frac{-1}{n}$$

and (6) survives unchanged because in the more general formulation, using (22) we have:

$$(6') \qquad \frac{\partial G_i^\lambda}{\partial R} = \frac{1}{\bar{y}}\left[\lambda \frac{\partial A_i}{\partial R} + (1-\lambda)\frac{\partial D_i}{\partial R}\right] = -\frac{1}{n\bar{y}}$$

Proposition 3 described the impact on G_i of making a small transfer from another person to individual i. Assuming $y_1 < y_2 < \cdots < y_n$, the conclusion in the λ = ½ case was that, except for a very small region around the median, a transfer from a higher income person would reduce G_i if y_i was below the median, and increase G_i if y_i was above the median. Converse results held if the transfer came from a lower income person. The critical role of the median arose because with λ = ½, advantage, A_i, and deprivation, D_i, are equally weighted. In general, the critical percentile is given by 1-λ. Thus, for example if one placed half as much weight on A_i as on D_i, i.e. λ = 1/3, the critical percentile would be 2/3. That means that a small transfer from someone with higher income would be regarded as equalizing by almost everyone in the bottom two thirds of the population, but as disequalizing by almost all of those in the top third. This occurs because putting a higher weight on D_i increases the equalizing impact on G_i^λ from the fall in D_i caused by such a transfer.

V. Personal Inequality Assessments at Different Income Levels

In this section we examine how G_i^λ varies as y_i rises from y_1 to y_n. We provide results for the general case where λ can take on any value in the interval [0,1], but note specific conclusions for the case where λ = ½ .

How does G_i^λ change as we move up through the distribution of income? We continue to assume $y_1 < y_2 < \cdots < y_n$. As we go from individual i to i+1, the absolute income gaps in (3) or implicitly in (22) increase in value by $y_{i+1} - y_i$ for all j such that $y_j < y_i$, and the corresponding gaps for all $j > i$ fall by the same amount. Hence we should expect that G_i^λ will initially decline as i rises from 1, since at the start there are more people with $j > i$ than with $j \le i$, until some critical point is reached, beyond which G_i should begin to increase. Formally we have:

<u>Proposition 4:</u> If $y_1 < y_2 < \cdots < y_n$,

$$G_{i+1}^\lambda \;\substack{>\\=\\<}\; G_i^\lambda \quad \text{as} \quad \frac{i}{n} \;\substack{>\\=\\<}\; 1-\lambda\,.$$

Proof: See Appendix.

Proposition 4 indicates that G_i^λ falls up to the $(1-\lambda)100$th percentile of the distribution and increases above that. As indicated above, this U-shaped pattern is based on the fact that moving from income y_i to income y_{i+1} increases the income gaps with lower income people and reduces those with higher income people by the same absolute amount. The relative impact of changes in the upper gaps compared with that of changes in the lower gaps is (1-λ)/λ. This means that G_i^λ will fall more rapidly starting from i = 1 if λ < ½, compared with the λ = ½ case, and less rapidly if λ > ½. Note that if $\lambda = \frac{1}{2}$, $G_i^\lambda = G_i$ falls up to the 50th percentile, that is up to the median, and rises thereafter.

We can also readily identify the value of G_i^λ at the bottom and top of the distribution (i = 1 and i = n), as well as the value of G_i^λ for the median individual, G_{med}^λ, if n is odd. We have:

Proposition 5: If $y_1 < y_2 < \cdots < y_n$,

(i) $G_1^\lambda = (1-\lambda)(1-\frac{y_1}{\bar{y}})$

(ii) if n is odd, $G_{med}^\lambda = \frac{n-1}{2n\bar{y}}[(1-\lambda)\bar{y}_{med}^h - \lambda\bar{y}_{med}^l]$; if n is even, G_{med}^λ is not defined,

(iii) $G_n^\lambda = \lambda(\frac{y_n}{\bar{y}} - 1)$

Proof: See Appendix.

Proposition 5 allows us to put upper bounds on G_1^λ and G_n^λ. If y_1 is non-negative, the highest possible value of G_1^λ is $1-\lambda$, which occurs when $y_1 = 0$. When individuals weight deprivation and advantage equally, that is when $\lambda = \frac{1}{2}$, the maximum value is $\frac{1}{2}$. But the maximum value of G_1^λ ranges from 0, when $\lambda = 1$ and people care only about advantage, to 1 when $\lambda = 0$ and people only care about deprivation. In view of Proposition 4, these maxima also apply to all G_i^λ up to the $(1-\lambda)100$th percentile.[7] The upper bound on G_n^λ occurs when one individual has all the income and $y_n = n\bar{y}$. In that case $G_n^\lambda = \lambda(n-1)$, which is also an upper bound for all G_i^λ's above the $(1-\lambda)100$th percentile.

Part (ii) of the proposition is also interesting, in throwing light on the value of the personal inequality index for the "average person", that is on the value of G_{med}^λ. The latter is based on a weighted average of $\bar{y}_{med}^h$ *and* $\bar{y}_{med}^l$, with the weight on $\bar{y}_{med}^h$ falling with λ. In the focal case with $\lambda = 1/2$, we have:

$$G_{med} = \frac{(n-1)}{4n\bar{y}}(\bar{y}_{med}^h - \bar{y}_{med}^l)$$

Since in any real-world example $(n-1)/n \approx 1$, this says:

$$G_{med} \approx \frac{\bar{y}_{med}^h - \bar{y}_{med}^l}{4\bar{y}}$$

In the U.S. today, for household income before tax, $\bar{y}_{med}^h \approx \frac{8}{5}\bar{y}$ and $\bar{y}_{med}^l \approx \frac{2}{5}$, which yields $G_{med} \approx 0.3$, less than the value of the Gini coefficient, which was 0.476 in 2013.[8] We may also note values of G_{med} under some familiar continuous distributions. G_{med} would equal $\frac{1}{4}$ for a uniform distribution, and if $y_i \sim N(\mu, \sigma)$, it would equal $\frac{2}{5}\frac{\sigma}{\mu}$, that is two-fifths of the coefficient of variation.

[7] Note that with $\lambda = 1$, the $(1-\lambda)100$th percentile = 0, so that G_i^λ has no falling range.

[8] With the help of quintile share and other data from U.S. Census Bureau (2015) it can be estimated that $\bar{y}_{med}^h = 1.64\bar{y}$ and $\bar{y}_{med}^l = 0.36\bar{y}$.

We can see that G_i^λ will generally not be symmetric around the median. Looking at the $\lambda = 1/2$ case again, for example, G_i will never be greater than $1/2$ at the lowest income level, but can be very high at the top end. G_i is not bounded above by 1, unlike the Gini coefficient. $G_n = 1$ is reached when $\frac{y_n}{\bar{y}} = 3$. That ratio is exceeded in almost all real-world cases. This implies that, in a mathematical sense, the rich perceive that there is more inequality than do the poor when $\lambda = 1/2$, which is not unintuitive. If you are rich there are relatively few people whose incomes are close to yours, meaning there is a large gulf between your income and most others'.

VI. Personal Inequality During Secular Change in Income Distribution

This section asks how G_i^λ can be predicted to behave at different income levels during periods of secular change in income distribution. We focus initially in each case on the $\lambda = 1/2$ case, in which individuals weight deprivation and advantage equally, referring to $G_i^{1/2}$ simply as G_i, as above. We start with the Kuznets transformation and go on to the polarization and rising inequality that has been seen in the U.S. and many other high income countries in the last few decades. The principles at work are explored with the help of examples, which are intended merely to be illustrative.

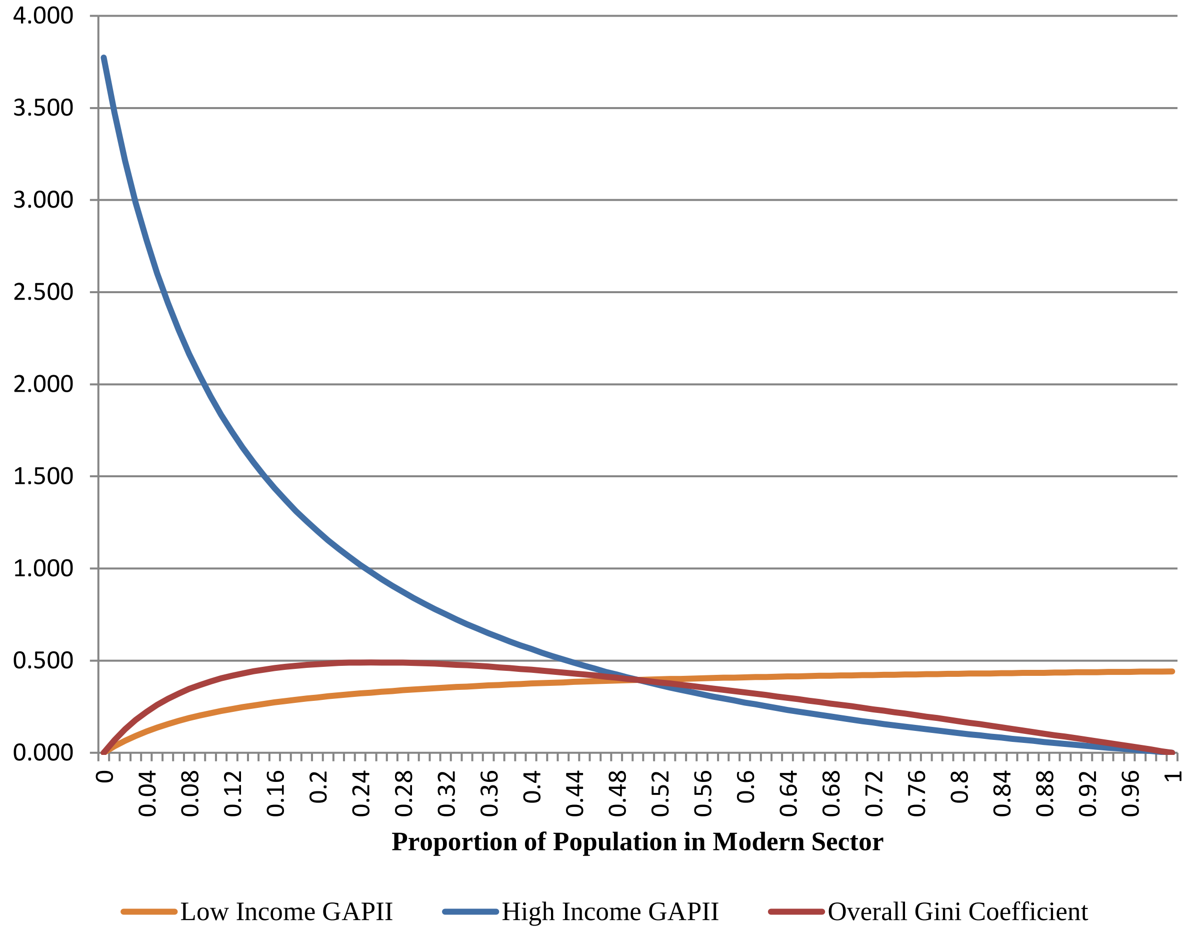
Kuznets Transformation

Kuznets (1955) studied what happens to income distribution and inequality in a growing economy where the composition of output is shifting from an initially large traditional agricultural sector to a modern sector. The modern sector eventually comprises most if not all of the economy. The consequences for inequality can be illustrated by considering a stylized model in which individual incomes are uniform within each of the sectors, higher in the modern sector, and unchanging during the growth process.[9] In this case the Gini coefficient, *G*, rises until the fraction of the population in the modern sector, *p*, hits a critical value, after which it declines. This critical value of *p* is less than one half. That is because, while the mean difference has a maximum at $p = 1/2$, the mean, which appears in the denominator of the expression for *G*, is rising throughout, so *G* has already started to decline at $p = 1/2$.

The behavior of the GAPIIs, that is the G_is, and *G* during the Kuznets transformation will be illustrated here using an example whose implications are shown in Figure 1. It is assumed that income of each person in the traditional sector is 11.7% of per capita income in the modern sector. This gap is sufficient for the peak value of *G* to be 0.49, the value observed in China in 2008 (Li and Sicular, 2014). China is the most prominent recent example of a society going through the kind of transformation that Kuznets described. In the early 1980s its Gini coefficient for family income fluctuated around 0.30 (Sicular,

[9] Kuznets considered a richer range of possibilities. He allowed unequal income distribution within both sectors and believed the leading case was one in which there was greater inequality in the modern sector than in the traditional, or agricultural, sector. He also considered the impacts of changes in the relative income, and of income inequality, in the modern vs. the agricultural sector over time. In most cases he found that as the relative population of the agricultural sector declined over time there was an initial increase in inequality followed by a decline.

Figure 1: GAPIIs and Overall Gini Coefficients -- Kuznets Curve Case



2013). This was followed by a rapid rise with later deceleration to the 2008 peak, after which G began to fall slowly. [10] China may now be past a Kuznets curve peak..[11]

We will refer to the individual inequality measures of people in the low and high income groups as G_L and G_H respectively. Since no one is worse off than those in the low income group, $G_L = \frac{D_L}{\bar{y}}$, that is it is based entirely on deprivation, while $G_H = \frac{A_L}{\bar{y}}$ and is based wholly on advantage. As shown in Figure 1, when the modern sector is tiny, G_L is not far above zero. Almost everyone in the society has the same low income, so that n_L^h/n and therefore D_L are very low. The situation in the modern sector is the opposite. Since almost everyone has much lower income than those in the modern sector, the individual inequality measure there, G_H is very high. Now, as development proceeds, G_L rises monotonically and

[10] The National Bureau of Statistics estimates of the national Gini coefficient for family income were 0.491 in 2008 (Li and Sicular, 2014, Appendix A) and 0.469 in 2014 (Qi, 2015).

[11] Knight (2014) discusses whether China may be beyond the peak of the Kuznets curve. His conclusion is that this depends in part on public policy but that there are now strong underlying forces pushing in the direction of reducing inequality in China.

G_H falls monotonically (and dramatically, in the example reflected in Figure 1). It is as if people in the traditional sector become steadily more aware of the inequality between themselves and people in the modern sector as the modern sector grows. On the other hand, from the viewpoint of individuals in the modern sector, inequality is falling because more and more of their fellow citizens are as well off as they are.

How does one resolve the conflict when two population groups have such radically opposed views about the trend in inequality? The Gini coefficient offers a solution - - take an average of the individual assessments. Thus in the Kuznets curve example, G is a population weighted average of the values of G_L and G_H. An alternative would be to take a vote on the question of whether inequality was rising or falling - - a "democratic" approach. Here the democratic approach would say that inequality rises until $p = ½$ and falls thereafter. In the example, G says that inequality rises until $p = ¼$ and falls thereafter. That is because G_H falls faster than G_L rises, so that averaging G_H and G_L, even using population weights, places greater relative importance on the decline in G_H than on the rise in G_L. Thus the Gini procedure of averaging individual inequality assessments does not correspond to the democratic approach in this situation, and places more importance on the views of the wealthy.

Our analysis shows that, unfortunately, use of the Gini coefficient could cause confusion about what is happening to inequality during the Kuznets transformation due to its greater sensitivity to the views of the high income group. The Gini begins to fall "too soon". If the behavior of G were used as an input into policy decisions, this could potentially lead to a relaxation of inequality-reducing measures in a country where the majority of the population had yet to join the modern sector and still felt that inequality was rising.

The above analysis would not be affected significantly by moving from the $\lambda = ½$ case to $\lambda \neq ½$. There would of course be no impact on the time path of G. Since those in each sector are only concerned either about deprivation (in the traditional sector) or advantage (in the modern sector) what occurs at the individual level is simply a rescaling of G_L and G_H at each point in the Kuznets process. A majority of people still believe inequality is rising until $p = ½$ is reached, and above this point the majority think inequality is falling. G still has its peak at the same point as with $\lambda = ½$. In terms of Figure 1, there will be a proportionate shift of the G_L curve by the factor $2(1-\lambda)$ and a shift of the G_H curve in the opposite direction by the factor 2λ. In the case where $\lambda < ½$ the G_L and G_H curves will move towards each other, while if $\lambda > ½$ the result will be the opposite.

Polarization

There is much theoretical and empirical literature on polarization (including Foster and Wolfson, 1992; Esteban and Ray, 1994; Acemoglu and Autor, 2011; Autor and Dorn, 2013; Green and Sand, 2015). Polarization can take different forms. Without saying so, we have already been discussing one form in the context of the Kuznets transformation, which has two poles: the traditional society and the modern sector. At the starting point, with everyone in the traditional sector, there is no polarization. As population shifts to the modern sector polarization initially rises, as does aggregate inequality according to the Gini coefficient, which people in the traditional sector agree with but people in the modern sector do not. Then there is a phase where polarization continues to rise but changes in the Gini coefficient turn from positive to negative. Finally, when the modern sector population becomes a majority, polarization begins to fall, reaching zero when everyone is in the modern sector.

The behavior of polarization, the aggregate Gini coefficient, and personal inequality assessments over the course of the Kuznets transformation demonstrate that polarization may move in the opposite direction from both individual inequality assessments and an aggregate inequality measure. It is thus clear from the Kuznets case alone that the relationship between polarization and inequality is complex.

The relationship is even more complex in the case of the polarization in labor markets that has received attention in the US and other high income countries in recent years. In this case the relative demand for labor shifts away from mid-level occupations to both low-skilled and (especially) high skilled occupations Other things constant this should result in a shift in labor force composition away from the middle earning levels toward both high and low labor incomes. Such a shift has indeed occurred over significant timespans in the U.S., Canada, the UK, Germany and some other European countries (Acemoglu and Autor, 2011; Green and Sand, 2015). In most cases the relative wages of highly skilled workers have increased. In the US it has also been found that the relative wages of certain low skilled occupations have risen (Autor and Dorn, 2013).

We will analyze the kind of polarization seen over the last few decades in labor markets by first considering the effects of *population shift,* that is a rise in the number of individuals at low and high incomes combined with a reduction in the number at middle income. Subsequently we will look at the effect of changes in relative incomes as well. As in the Kuznets analysis it helps to consider a stylized situation. Assume that there are just three income levels in a society and that they display $y_L < y_M < y_H$. Numbers of individuals in the three groups are $n_L, n_M,$ and n_H. As in the Kuznets case the GAPIIs of people in the bottom group and top groups are given by $G_L^\lambda = \frac{\lambda D_L}{\bar{y}}$ and $G_H^\lambda = \frac{(1-\lambda) A_H}{\bar{y}}$.

Also as in the Kuznets analysis the increase in n_H will tend to make A_H and G_H^λ increase since (from 5i):

$$A_H = \frac{(n_L+n_M)}{n}(y_H - \bar{y}_H^l) \tag{23}$$

However, there is now an offsetting effect because $\bar{y}_H^l$ falls due to the population shift from the middle to lower groups, and therefore $(y_H - \bar{y}_H^l)$ increases. It can readily be shown that:

$$\Delta A_H, \Delta G_H \begin{array}{c} > \\ = \\ < \end{array} 0 \quad \text{as} \quad \frac{\Delta n_L}{-\Delta n_M} \begin{array}{c} > \\ = \\ < \end{array} \frac{y_H - y_M}{y_H - y_L} \tag{24}$$

Now $\frac{y_H - y_M}{y_H - y_L} < 1$ and $\frac{\Delta n_L}{-\Delta N_M} < 1$ as well, so it is not immediately clear which way the inequality will go. However, with a positively skewed distribution of income we would have $\frac{y_H - y_M}{y_H - y_L} > \frac{1}{2}$, so that if half or fewer of those leaving the middle income group go to the lower group (which is in line with the experience in the US at least), A_H and G_H will decline, as in the Kuznets case.

Turning to the bottom group, from (5ii) we have:

$$D_L = \frac{(n_M+n_H)}{n}(\bar{y}_L^h - y_L) \tag{25}$$

And it can be shown that:

$$(26) \qquad \Delta A_L, \Delta G_L \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{as} \quad \frac{-\Delta n_M}{\Delta n_H} \begin{matrix} > \\ = \\ < \end{matrix} \frac{y_H - y_L}{y_M - y_L}$$

Now, $\frac{y_H-y_L}{y_M-y_L} > 1$ and $\frac{-\Delta n_M}{\Delta n_H} > 1$ as well, so again there is ambiguity. Once more appealing to positive skewness, $\frac{y_H-y_L}{y_M-y_L} > 2$ is likely. So if half or more of those leaving the middle group go to the top group (which is of course the same as saying that half or fewer go to the bottom group, as above), A_L and G_L will fall, which is the opposite of what we found in the Kuznets analysis. This would be the result of the increase in $\bar{y}_L^h$ having a larger effect on A_H and G_H than the decline in $n_L^h = (n_M + n_H)$.

The analysis of A_L and A_H is sufficiently complex that one may (correctly) anticipate that the analysis of A_M and G_M would be tedious. This is not only because G_M depends on both A_M and D_M , immediately doubling the algebra, but also because for a general analysis allowing $\lambda \neq \frac{1}{2}$ one would need to think about how different weightings of A_M and D_M would affect the results. Clearcut results are hard to get. Suffice it to say that during polarization, population shift alone may increase, decrease, or leave unchanged personal inequality as viewed by the middle group.

What about changes in relative incomes associated with polarization? A robust finding across countries is that the relative income of the highly skilled has risen during observed labor market polarization. With the incomes of the two lower groups assumed unchanged, from (23) we see that this makes it more likely that A_H and G_H would rise with polarization, rather than declining as in the Kuznets analysis. This effect would be strengthened if $\bar{y}_H^l$ declined, which could occur as a result of y_M falling, which is also consistent with what is generally observed. From (25) we can see that a rise in y_H could also give A_L and G_L more of a tendency to increase, via its effect on $\bar{y}_L^h$, although that could be offset by a fall in y_M which would act to reduce $\bar{y}_L^h$.

Given the theoretical ambiguity of the behavior of $G_L, G_M,$ and G_H it is helpful to consider an example based on real-world observations. Autor and Dorn (2013) set out the changes in employment shares and wage rates for six broad occupational groups in the U.S. from 1980 to 2005. As shown in Table 1 here, the top group, consisting of managers, professionals, technicians, finance and public safety occupations experienced a 29% increase in employment share and a 36% rise in wage rates over those years. The middle group shown in Table 1, which aggregates the middle four occupational groups in Autor and Dorn (2013), had a 22% drop in employment share and only a 9% increase in wages. Finally, the bottom group, consisting of service occupations, had a 30% rise in employment share and a 17% increase in wages. These changes provide a dramatic example of labor market polarization.

Table 1 shows G_i rising for all three groups, as do A_i and D_i where applicable. The wage gap between the top group and the rest of the labor force expands considerably, leading to A_H more than doubling from 1980 to 2005. The middle group experiences a large increase in deprivation, which is not surprising in view of its poor wage performance and the large employment and wage increases for the top group. But the middle group also sees a rise in its advantage over the bottom group, which is due to the increase in the relative size of the latter group. The 17% wage rise of the bottom group is not large enough to

Table 1
Advantage, Deprivation and Personal Inequality Indexes with λ = 1/2, by Occupation Group - - Polarization Example Based on U.S. Data, 1980 and 2005

I. 1980

Occupation Group	Employment Share	Mean Wage (2004 $s)	A_i	D_i	G_i
1. Top	0.316	17.0	3.42	0	0.126
2. Middle	0.585	12.6	0.44	1.38	0.067
3. Bottom	0.099	8.2	0	5.36	0.198

II. 2005

Occupation Group	Employment Share	Mean Wage (2004 $s)	A_i	D_i	G_i
1. Top	0.409	23.1	6.11	0	0.226
2. Middle	0.462	13.7	0.53	3.86	0.162
3. Bottom	0.129	9.6	0	7.41	0.274

Notes: (i) The mean wage is the geometric mean hourly wage, derived from the mean log hourly wage reported by Autor and Dorn (2013).
(ii) The Top occupational group is the first category in Autor and Dorn (2013). It includes managers, professionals, technicians, finance and public safety occupations. The Middle occupational group consists of the four middle groups in Autor and Dorn (2013): production and craft occupations; transportation, construction, mechanics, mining and farm occupations; machine operators and assemblers; and clerical and retail sales occupations. The Bottom occupational group consists of service occupations.
(iii) $A_i, D_i,$ and G_i are the personal advantage, deprivation, and inequality indexes.
Source: Employment share and mean wage are from Autor and Dorn (2013, Table 1) - - see Note (i). The other columns are from calculations by the author.

overcome the deprivation-increasing effect for it of the expansion and large wage increase of the top group, so its deprivation increases quite a bit.

The above results are obtained with $\lambda = 1/2$, of course. But changing λ will not produce a direction of change in G_i^{λ} different from that in G_i, since we do not have a case where either advantage or deprivation are falling. Reweighting A_i and D_i cannot produce a sum that decreases. This result does, however, depend on how Autor and Dorn's original six broad occupational groups are aggregated into three groups.

Autor and Dorn (2013) stress that the only low wage group that sees a rise in employment share is their service occupations group, and that original group has been treated here as the bottom of our three more aggregated groups. However, although the original 1980 group with the second-lowest wage, those in clerical and retail sales occupations, has a small drop in employment share, it, like the service occupations, shows a relative wage *increase*. Thus the clerical and retail sales occupations benefit from wage polarization if not from employment polarization. Again aggregating to three groups, but putting clerical and retail sales occupations in the bottom category along with the service occupations, changes results a little. $G_L, G_M,$ and G_H all increase, but A_M falls. Hence, if λ is sufficiently high, more precisely 0.72 or more, G_M *declines* between 1980 and 2005. That is, if the middle group is sufficiently concerned about advantage it will regard polarization as having reduced inequality in this case. While worth noting, this result may not affect one's conclusions much in view of the broad consensus in the literature that it is likely that λ ≤ ½.

Rising Overall Inequality

In the last four decades substantial periods of rising overall income inequality, as measured by the Gini coefficient and other conventional indexes, have been observed in a wide range of high income countries (Roine and Waldenström, 2015). In some cases this reflects polarization, but labelling all increases in inequality as polarization would abuse the latter term. It is probably best to refer to a broadly-based downward movement in the Lorenz curve simply as an increase in inequality.

Table 2

Personal Inequality Indexes with $\lambda = \frac{1}{2}$, by Overall Gini Coefficient and Percentile, Lognormal Example

Overall Gini:	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
Percentile:					
1	0.297	0.380	0.434	0.467	0.486
5	0.244	0.328	0.392	0.437	0.468
10	0.213	0.295	0.362	0.414	0.452
25	0.165	0.240	0.307	0.366	0.415
50	0.140	0.208	0.272	0.331	0.385
75	0.175	0.261	0.342	0.415	0.476
90	0.273	0.427	0.586	0.743	0.886
95	0.362	0.591	0.850	1.135	1.431
99	0.585	1.048	1.670	2.502	3.608

It is interesting to ask what is likely to happen to personal inequality assessments during a period of rising inequality. Table 2 provides some insight on this question. Using $\lambda = 1/2$, it shows G_i at selected percentiles of lognormal income distributions that have overall G = 0.2, 0.3, 0.4, 0.5 and 0.6. These Gini values span most of the range observed across countries. For reference, the Gini coefficient for household income in the U.S. was 0.397 in 1975 and rose with little interruption to 0.476 in 2013 (U.S. Census Bureau, 2015). In the UK the Gini coefficient for equivalized household income was 0.272 in 1977 and rose to 0.324 in 2013/14 (Office of National Statistics, 2015, Figure 5).

Table 2 shows, first, that G_i falls with income up to the median and then rises, as predicted by Proposition 4. The latter increase, from percentile to percentile, rises with income, particularly at the highest levels. We see, for example, with an overall Gini of 0.4 that while G_i approximately doubles, from 0.272 to 0.586, in going from the median to the 90th percentile, it then roughly triples to arrive at 1.670 for the 99th percentile. Second, the table shows that sensitivity to rising inequality is greatest at top income levels. This is more clearly illustrated in Table 3 which shows high income-low income G_i ratios by percentile, given different values of G. The P90:P10 G_i ratio rises from 1.62 when G = 0.4 to 1.80 when G = 0.5, and the P99:P1 ratio rises from 3.85 when G = 0.4 to 5.36 when G = 0.5. The increase of G by 0.1, from 0.4 to 0.5 is roughly similar to the rise seen in the U.S. since 1975, so this is suggestive with respect to real-world changes in inequality assessments by people at the top of the income distribution. Thus these results raise the interesting possibility that high income people could have experienced the largest perceived increases in inequality in recent decades.

Table 3

Ratios of Personal Inequality Indexes for Selected Percentiles with $\lambda = 1/2$, Lognormal Example

Overall Gini:	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
P60/P40	1.004	1.006	1.007	1.008	1.009
P70/P30	1.030	1.044	1.056	1.065	1.070
P75/P25	1.059	1.087	1.112	1.132	1.146
P80/P20	1.103	1.155	1.204	1.247	1.277
P90/P10	1.281	1.444	1.618	1.795	1.960
P95/P5	1.483	1.799	2.171	2.596	3.060
P99/P1	1.968	2.757	3.850	5.361	7.430

An idea of the quantitative impact of allowing $\lambda \neq 1/2$ is provided in Tables 4 and 5, which repeat the exercises of Table 2 and 3, but with the range of G confined to [0.3,0.5] and alternate values of λ = 0.25 and 0.75 considered. Note first that G_i^λ initially declines as income rises but hits a minimum at the $(1-\lambda)100$th percentile , as predicted by Proposition 4. Next, we can see that raising λ twists the G_i^λ profile. For lower incomes, G_i^λ falls but for higher incomes G_i^λ rises. This means that there is an increase with λ in the acceleration of G_i^λ as one goes up the income scale, and a rise in G_i^λ ratios for such income percentiles as P90/P10 and P99/P1 (Table 5). The switch from a negative to positive impact of λ on G_i^λ occurs at P61 for G = 0.3, at P65 for G = 0.4, and at P69 for G = 0.5.

Table 4

Personal Inequality Indexes by Overall Gini Coefficient and Percentile with Alternative Values of λ, Lognormal Example

	λ = 0.25			**λ = 0.75**		
Overall Gini:	**0.3**	**0.4**	**0.5**	**0.3**	**0.4**	**0.5**
Percentile:						
1	0.570	0.650	0.700	0.190	0.217	0.233
5	0.491	0.586	0.655	0.166	0.197	0.219
10	0.439	0.539	0.618	0.152	0.185	0.210
25	0.341	0.443	0.533	0.139	0.172	0.199
50	0.242	0.332	0.423	0.173	0.211	0.240
75	0.199	0.278	0.361	0.322	0.406	0.468
90	0.242	0.342	0.449	0.611	0.830	1.037
95	0.311	0.452	0.613	0.870	1.248	1.656
99	0.528	0.843	1.266	1.567	2.496	3.739

Note: λ is the weight placed on relative advantage, A_i, in the calculation of the personal inequality index, G_i. See text.

Table 5

Ratios of Personal Inequality Indexes for Selected Percentiles with Alternative Values of λ, Lognormal Example

	λ = 0.25			**λ = 0.75**		
Overall Gini:	**0.3**	**0.4**	**0.5**	**0.3**	**0.4**	**0.5**
P60/P40	0.787	0.811	0.837	1.406	1.401	1.382
P70/P30	0.636	0.674	0.717	1.965	1.974	1.949
P75/P25	0.584	0.627	0.677	2.320	2.358	2.353
P80/P20	0.549	0.599	0.656	2.747	2.847	2.897
P90/P10	0.552	0.634	0.727	4.016	4.487	4.941
P95/P5	0.633	0.772	0.936	5.252	6.329	7.548
P99/P1	0.927	1.296	1.808	8.239	11.506	16.015

Note: λ is the weight placed on relative advantage, A_i, in the calculation of the personal inequality index, G_i. See text.

VII. Discussion and Conclusion

Recognizing the Gini coefficient as the average of "Gini admissible" personal inequality indexes or GAPIIs generates rich results. One remarkable feature is that GAPIIs are completely insensitive to transfers of income that occur only among people who have incomes above those of the reference individual, or among those with incomes below. This means that GAPIIs do not obey the Pigou-Dalton principle of transfers. But they do regard transfers from those in the "above group" to those in the "below group" as equalizing and transfers in the other direction as disequalizing. These properties help to explain why the Gini coefficient's sensitivity to transfers depends critically on the number of people with incomes between those of the donor and recipient. That property can now be seen to result from the fact that the only transfers an individual with a GAPII regards as affecting inequality at all, aside from those that alter their own incomes, are transfers that "pass over" them.

As we have seen, each GAPII is a weighted average of an individual's deprivation and advantage. That the relative weights placed on these components can vary has a range of implications. For example, the weights could vary across societies. In one society people might care only about deprivation - - "resenting" the fact that others have higher incomes. At the other extreme, in another society, they might only care about advantage - - showing concern for the "less fortunate". And, of course, any weighting between these extremes may be allowed. But in each society, taking the average of GAPII values would yield the Gini coefficient. So societies that have quite different views about assessing inequality at the individual level may still all embrace the Gini coefficient as their aggregate measure of inequality. This conclusion may help to justify the globally widespread use of the Gini coefficient in both applied work and popular literature.

The pattern of GAPII values found as we go up the income scale is also of interest. Starting from the lowest income, the personal index values fall up to a point - - the median when deprivation and advantage are weighted equally - - and then rise. With the positively skewed income distributions seen in the real world, if deprivation is not weighted sufficiently less than advantage, the value of the index will rise to a higher level at top incomes than it displays at low income levels. This asymmetry means that, as the average of individual GAPIIs, the Gini coefficient may be more sensitive to the views of higher income individuals than to those of lower income people on a person-by-person basis.

The paper has also discussed how personal inequality assessments may behave during periods of secular change in income distribution. In the development context we have seen that, in the simplest model, people in the traditional sector will regard inequality as rising *throughout* the Kuznets transformation, while those in the modern sector think precisely the opposite. The resulting scope for misunderstanding and conflict seems large. This may throw some light on the tensions that are observed during periods of rapid modernization in developing countries. A further insight comes from the fact that the Gini coefficient says the Kuznets process stops being disequalizing well before half the population is in the modern sector. Thus, the Gini is not always a guide to majority opinion.

Less clearcut results were obtained for polarization. Under polarization, population shifts not only to the top but also to the bottom, with a shrinking middle group. Complex changes in relative incomes can also occur. The result is that there are circumstances under which people in each of the top, middle and bottom income groups may regard inequality as rising, and others in which they may all think it is falling, or may have mixed assessments. Given this ambiguity we turned to the real world for some guidance. In a three-group example set up to parallel the actual polarization seen in the U.S. over the period 1980 – 2005, personal inequality rose from the viewpoint of all three groups in a base case. However, broadening the bottom group led to the result that the middle group could have regarded inequality as falling if it placed a sufficiently high weight on advantage compared with deprivation.

Finally, we examined the impact of a general spreading of the income distribution by seeing how rising dispersion of a lognormal distribution would affect personal inequality assessments. Such a trend raises personal inequality values at all levels of income irrespective of the relative weights placed on deprivation and advantage. However it does not do so equally. Unless sufficiently more weight is placed on deprivation, the increase in inequality is greatest from the viewpoint of those with the highest incomes.

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Appendix

This appendix provides proofs of propositions 4 and 5.

Proposition 4: If $y_1 < y_2 < \cdots < y_n$,

$$G_{i+1}^{\lambda} \overset{>}{\underset{<}{=}} G_i^{\lambda} \quad \text{as} \quad \frac{i}{n} \overset{>}{\underset{<}{=}} 1-\lambda \, .$$

Proof: From (4′), (5) and (22), and using the assumption that $y_1 < y_2 < \cdots < y_n$, $G_{i+1}^{\lambda} \overset{>}{\underset{<}{=}} G_i^{\lambda}$ as:

$$\text{(A1)} \qquad \lambda \sum_{j=1}^{i} (y_{i+1} - y_j) + (1-\lambda) \sum_{j=i+2}^{n} (y_j - y_{i+1}) \overset{>}{\underset{<}{=}} \lambda \sum_{j=1}^{i-1} (y_i - y_j) + (1-\lambda) \sum_{j=i+1}^{n} (y_j - y_i)$$

Now, the left-hand side of this expression can be written:

$$\text{(A2)} \qquad \lambda\left[\textstyle\sum_{j=1}^{i-1} (y_i - y_j) + i(y_{i+1} - y_i)\right] + (1-\lambda)\left[\textstyle\sum_{j=i+1}^{n} (y_j - y_i) + (n-i)(\, y_i - y_{i+1})\right]$$

Hence, (A1) simplifies to:

$$\lambda \, i(y_{i+1} - y_i) + (1-\lambda)(n-i)(y_i - y_{i+1}) \overset{>}{\underset{<}{=}} 0$$

which is equivalent to:

$$\lambda \, i - (1-\lambda)(n-i) \overset{>}{\underset{<}{=}} 0$$

which becomes:

$$\lambda n + i - n \overset{>}{\underset{<}{=}} 0$$

from which one readily derives the result that

$$G_{i+1}^{\lambda} \overset{>}{\underset{<}{=}} G_i^{\lambda} \quad \text{as} \quad \frac{i}{n} \overset{>}{\underset{<}{=}} 1-\lambda$$

Proposition 5: If $y_1 < y_2 < \cdots < y_n$,

(i) $G_1^\lambda = (1-\lambda)(1-\frac{y_1}{\bar{y}})$

(ii) if n is odd, $G_{med}^\lambda = \frac{n-1}{2n\bar{y}}[(1-\lambda)\bar{y}_{med}^h - \lambda\bar{y}_{med}^l]$; if n is even, G_{med}^λ is not defined,

(iii) $G_n^\lambda = \lambda(\frac{y_n}{\bar{y}} - 1)$

Proof: (i) From (5) and (22), given that $n_1^l = 0$,

$$G_1^\lambda = \frac{(1-\lambda)}{n\bar{y}}[n_1^h(\bar{y}_1^h - y_1)]$$

$$= \frac{(1-\lambda)}{n\bar{y}}[\sum_{j=2}^{n} y_j - (n-1)y_1]$$

$$= \frac{(1-\lambda)}{n\bar{y}}(\sum_{j=1}^{n} y_j - ny_1)$$

$$= \frac{(1-\lambda)}{n\bar{y}}(n\bar{y} - ny_1)$$

$$= (1-\lambda)(1-\frac{y_1}{\bar{y}})$$

(ii) From (5) and (22), if n is odd we have:

$$G_{med}^\lambda = \frac{\lambda}{n\bar{y}} n_{med}^l\left(y_{med} - \bar{y}_{med}^l\right) + \frac{(1-\lambda)}{n\bar{y}} n_{med}^h\left(\bar{y}_{med}^h - y_{med}\right)$$

Noting that $n_{med}^l = n_{med}^h = \frac{n-1}{2}$,

$$G_{med}^\lambda = \frac{n-1}{2n\bar{y}}[(1-\lambda)\bar{y}_{med}^h - \lambda\bar{y}_{med}^l]$$

If n is even there is no individual with median income since $y_1 < y_2 < \cdots < y_n$.

(iii) From (5) and (22), given that $n_n^h = 0$,

$$G_n = \frac{\lambda}{n\bar{y}}[n_n^l\left(y_n - \bar{y}_n^l\right)]$$

$$= \frac{1}{n\bar{y}}[(n-1)y_n - \sum_{j=1}^{n-1} y_j]$$

$$= \frac{\lambda}{n\bar{y}}[ny_n - \sum_{j=1}^{n} y_j]$$

$$= \frac{\lambda}{n\bar{y}}[ny_n - n\bar{y}]$$

$$= \lambda[\frac{y_n}{\bar{y}} - 1]$$